Invariant Distributions and Stationary Correlation Functions of One-Dimensional Discrete Processes

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The connection between one-dimensional dynamical laws generating discrete processes and their invariant densities as well as their stationary correlaton functions is discussed. In particular the changes occuring under a special equivalence transformation are considered. Correlation functions are used to describe the gradual transition from periodic states to chaotic states via periodic motions with superimposed nonlinearity noise.

I. Introduction

There are certain processes in physics, chemistry, ecology, etc. in which the time development is described by giving the values $x(t_\tau)$ of the dynamical variables at a discrete sequence of times t_τ. Instead of a differential equation of motion or its solution one considers some function $f(x)$ which generates the process by

$$x_{\tau+1} = f(x_\tau), \quad \tau = 0, 1, 2, \dots \quad (1)$$

In what follows we denote the τ^{th} iterate of $f(x)$ by

$$f^{[\tau]}(x) := f[f^{[\tau-1]}(x)], \quad f^{[0]}(x) := x. \quad (2)$$

If there are properties of the sequence $\{f^{[\tau]}(x_0)\}$ which do not depend on the initial conditions x_0, these invariants reflect characteristics of $f(x)$ alone.

In this paper we study for some dynamical laws $f(x)$ not only the asymptotic density functions they generate,

$$\varrho_f^*(x) = \lim_{\substack{m(I)\to 0\\ I\to x}} \lim_{N\to\infty} \frac{1}{m(I)} \frac{1}{N} \sum_{\tau=0}^{N-1} \chi_I[f^{[\tau]}(x_0)] \quad (3)$$

but also the time correlation of the process,

$$C_f(\tau) = \langle \delta x\, \delta x_\tau \rangle := \langle x f^{[\tau]}(x) \rangle - \langle x \rangle^2. \quad (4)$$

This latter is a useful means to understand why apparently stochastic processes may arise from simple deterministic dynamical laws. The mean $\langle \dots \rangle$ is either a time mean or, if f turns out to be ergodic, the ensemble mean with ϱ_f^*. $m(I)$ denotes the size of the interval I (Lebesgue measure) and χ_I its characteristic function (1 if x inside I, 0 elsewhere).

In Sect. II we cite two theorems of basic importance for our subject and give an ergodicity criterion for dynamical laws f based on a theorem by Li and Yorke. In Sect. III we discuss a method to compute ϱ_f^* and in IV we show how the set of accessible dynamical laws can be extended by introducing an equivalence relation. The computation of correlation functions in V leads us in VI to a further discussion of the previously mentioned equivalence relation. In VII we show that under certain well defined conditions the processes generated by nonlinear dynamical laws may be described as periodic motions in state-space superimposed by noise.

There is great recent interest in such "nonlinearity noise" in connection with experimental [1, 2] and theoretical [3, 4] studies of hydrodynamic turbulence, chemical turbulence [5], etc.

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II. Two Theorems on Asymptotic Distributions

In statistical physics dynamical laws f which are ergodic or even strongly mixing are of major importance. The existence of a unique L^1-integrable invariant density ϱ_f^* generating an absolutely continuous measure

$$\mu_f^*(I) = \int_I \varrho_f^*(x)\, \mathrm{d}x \quad (5)$$

is equivalent to the ergodicity of f [6, 7]. Lasota, Li and Yorke proved the following two theorems on the existence and uniqueness of such an ϱ_f^* [8].

T1: *Existence* [9]

Let f: $[0, 1] \to [0, 1]$ be a piecewise C^2-function such that

$$\inf_x \left| \frac{\mathrm{d}f^{[n_0]}(x)}{\mathrm{d}x} \right| > 1 \quad (6)$$

for one positive integer n_0. Then there is a function $\varrho_f^*(x)$ with the following properties:

(a) $\varrho_f^* \geqq 0$,

(b) $\int_0^1 \varrho_f^*(x)\,\mathrm{d}x = 1$,

(c) ϱ_f^* is invariant under f, i. e.

$$\varrho_f^*(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{f^{-1}([0,x])} \varrho_f^*(x')\,\mathrm{d}x' .$$

The main purpose of the infimum-condition (6) is to exclude the stability of possibly existing periodic points

$$x^* = f^{[\tau]}(x^*), \quad \tau = 1, 2, 3, \ldots .$$

It implies a severe restriction on the applicability of the theorem since e. g. all continuous differentiable functions on $[0,1]$ are excluded. In IV we will see that this restriction can, at least partially, be overcome.

The next theorem gives some insight into the structure of the set R_f of invariant densities of a dynamical law f from which the uniqueness of ϱ_f^* can be inferred:

T2: *Structure of* R_f [10]

Let f: $[0,1] \to [0,1]$ be a piecewise C^2-function such that

$$\inf_x \left| \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right| > 1 .$$

The set $J := \{y_0, y_1, \ldots, y_k\}$ of points, where $\mathrm{d}f(y)/\mathrm{d}y$ does not exist, be finite [11].

Then there is a finite collection of sets

$$M_1, M_2, \ldots, M_n$$

and a set R_f of L^1-functions of bounded variation, which are invariant under f, with a subset

$$\{\varrho_1, \varrho_2, \ldots, \varrho_n\}$$

such that

(a) each $M_i\,(1 \leqq i \leqq n)$ is a finite union of closed intervals;

(b) $M_i \cap M_j$ contains at most a finite number of points if $i \neq j$;

(c) each $M_i\,(1 \leqq i \leqq n)$ contains at least one point $y_j\,(1 \leqq j \leqq k)$ in its interior; hence $n \leqq k$;

(d) $\varrho_i(x) = 0$ for $x \notin M_i\,(1 \leqq i \leqq n)$ and $\varrho_i(x) > 0$ for almost all $x \in M_i$.

T2 immediately yields a first criterion of uniqueness, already formulated by Li and Yorke [10]:

C1: The invariant density ϱ_f^* of f is uniquely determined if f fulfills the propositions of T2 and has exactly one point of discontinuity in $(0,1)$.

This criterion implicitly contains a restriction to f with one hump because any further hump would imply a further point of discontinuity within $(0,1)$. Therefore we give a second criterion starting from the sets M_i of T2: Since M_i constitutes something like the closure of the support of ϱ_i, it is invariant under f up to a finite subset. So two different sets $M_i, M_j, i \neq j$ cannot be mapped on the same interval by any iterate of f. On the other hand, according to T2(c) each M_i contains a complete ε-neighbourhood of at least one point of discontinuity. This leads us to another uniqueness criterion:

C2: Let f satisfy the propositions of T2.

$$U_\varepsilon(y_j) := (y_j - \varepsilon, y_j + \varepsilon) \cap (0,1), \quad \varepsilon > 0$$

be the ε-neighbourhood of any point of discontinuity y_j. The invariant density ϱ_f^* is unique if there are positive integers n_1, n_2 for each pair (y_i, y_j) such that for arbitrarily small $\varepsilon > 0$

$$m\,(f^{[n_1]}[U_\varepsilon(y_i)] \cap f^{[n_2]}[U_\varepsilon(y_j)]) \neq 0 .$$

III. Computation of the Invariant Density ϱ_f^*

The computation of an invariant density ϱ_f^* for a given dynamical law f constitutes a problem which to the best of our knowledge has not yet been solved in complete generality. Usually one falls back to numerical methods. But even these, if the sequence $\{x_\tau\}$ is evaluated directly, are often extremely susceptible to round-off errors [12].

A numerically stable procedure yielding an approximation by step functions was proposed by Ulam [13]. The fundamental idea starts from the Frobenius-Perron operator $\hat{P}_f$, which describes the "time-evolution" of an arbitrary density ϱ_τ under f:

$$\varrho_{\tau+1}(x) = \hat{P}_f\,\varrho_\tau(x) := \frac{\mathrm{d}}{\mathrm{d}x} \int_{f^{-1}([0,x])} \varrho_\tau(x')\,\mathrm{d}x', \quad \varrho_\tau \in L^1 . \tag{7}$$

The desired invariant density ϱ_f^* obviously is a fixed point of $\hat{P}_f$. Ulam conjectured that $\hat{P}_f$ can be approximated by a matrix $\mathbb{P}$ with elements

$$p_{ij} = \frac{m[I_j \cap f^{-1}(I_i)]}{m(I_j)} . \tag{8}$$

$I_i\,(1 \leqq i \leqq n)$ are the elements of an arbitrary decomposition of $[0,\ 1]$. p_{ij} is that fraction of I_j

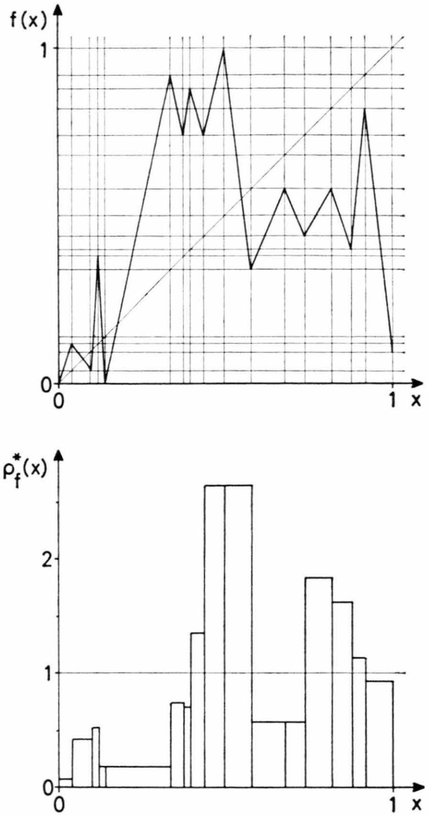

Fig. 1. The exact asymptotic distribution $\varrho_f^*(x)$ for a dynamical law f with finite K. If the sizes of the intervals I_i do not vary too much, the relative magnitude of ϱ_f^* in I_i can roughly be estimated by the number of branches of f mapping into I_i.

which is mapped into the interval I_i by $f(x)$. Assuming that $\hat{P}_f$ has exactly one fixed point ϱ_f^*, Ulam's conjecture was proved by Li [14], who showed that with increasing n the fixed point of $\mathbb{P}$ converges to ϱ_f^*.

It may be worthwhile to point out that for special classes of dynamical laws f one may even obtain an exact ϱ_f^* already for finite n. This holds for all broken linear transformations f satisfying T1 and C2 with a finite set $K := \bigcup_{\tau=0}^{\infty} f^{[\tau]}(J)$. One gets an analytic expression for ϱ_f^* from that $\mathbb{P}$ which particularly uses the decomposition of $[0, 1]$ generated by K. An example is given in Figure 1. Proof: If K is finite, each interval I_i of the decomposition is mapped linearly onto a union of such intervals. So all step functions being constant in each I_i are transformed by $\hat{P}_f$ into functions of the same type.

In these finite dimensional cases $\varrho(x)$ may be represented by a vector $\boldsymbol{r}$ with n components r_i defined by

$$r_i := \varrho(x \in I_i) \cdot m(I_i); \quad 1 \leqq i \leqq n, \tag{9}$$

the probability of I_i. ϱ_f^* corresponds to the eigenvector $\mathbb{P}\,\boldsymbol{r}_f^* = \boldsymbol{r}_f^*$ with eigenvalue 1. Since $\mathbb{P}$ is a stochastic, indecomposable matrix, the largest eigenvalues are of modulus 1 and the largest real eigenvalue is 1 [15]. So $\boldsymbol{r}_f^*$ can also be found by repeated multiplication of an arbitrary normalized vector $\boldsymbol{r}$ by $\mathbb{P}$:

$$\lim_{\tau \to \infty} \mathbb{P}^\tau \boldsymbol{r}_0 = \boldsymbol{r}_f^*$$

normalization condition: $\sum_{i=1}^{n} r_{0i} = 1$.

Finally we give some results concerning a special family of broken linear transformations. We consider

$$N_p(x) := (-1)^{[px]}\, p\, x \;(\mathrm{mod}\ 1), \quad x \in [0, 1], \quad p \in \mathbb{R}. \tag{10}$$

$[p\,x]$ is the integer part of $p\,x$. $N_p(x)$ is ergodic for $p > 1$. The reader may easily prove this statement by using T1 and C1 if $1 < p \leqq 2$ and applying C_2 if $2 < p$. Moreover the relations

$$N_p^{[\tau]}(x) = N_{p^\tau}(x), \quad p, \tau = 1, 2, 3, \ldots \tag{11}$$

and

$$\varrho^*_{N_p}(x) = 1, \quad p = 2, 3, 4, \ldots \tag{12}$$

hold, i. e., the sequence $\{x_\tau\}$ in the course of the physical process covers the interval $(0, 1)$ everywhere with equal weight.

IV. Dynamical Laws Related by Conjugation

As we already pointed out the propositions of T1 and T2 severely restrict the admitted dynamical laws f. To extend the range of validity of these theorems we make use of an equivalence relation described by Halmos [16] and by Ulam [17].

Two transformations f: $I \to I$ and g: $J \to J$ on intervals I and J are called conjugate if there exists a one-to-one map h: $I \xrightarrow{\text{onto}} J$ such that

$$g(x) = h\left(f[h^{-1}(x)]\right). \tag{13}$$

In what follows the conjugating function h will always be assumed to be continuous and sufficiently smooth.

h establishes a one-to-one correspondence between the number sequences $\{x_\tau\}$ and $\{\Theta_\tau\}$ generated by

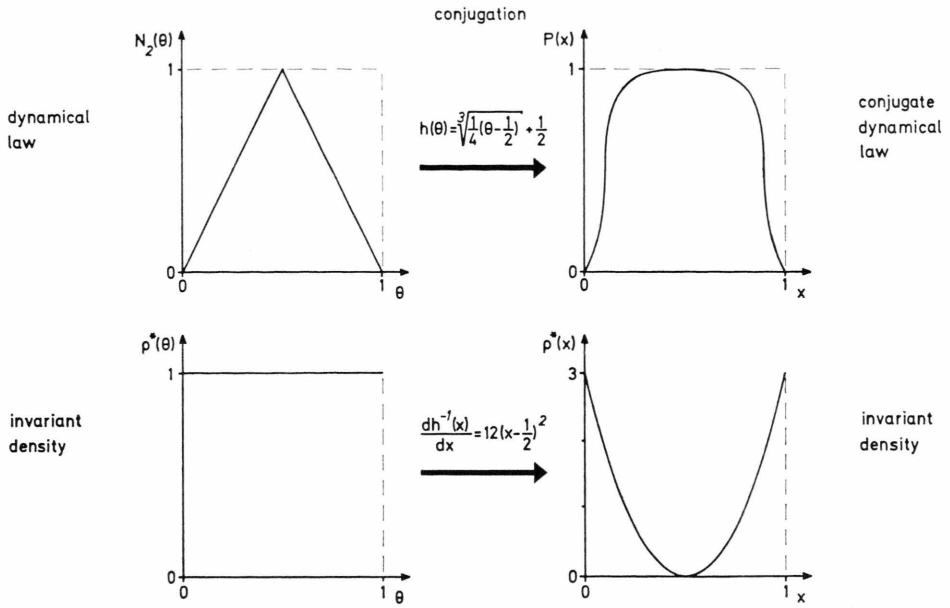

Fig. 2. Two conjugate dynamical laws and their corresponding invariant densities. $N_2(\Theta)$ satisfies the propositions of T1 and T2 but its derivate by conjugation, $P(x)=\sqrt[3]{0.125-2\,|x-0.5|^3}+0.5$, obviously does not. Nevertheless T1 and T2 can be applied to $P(x)$ due to the conjugation. The corresponding transformation of the density is controlled by the first derivative of $h^{-1}(x)$.

$x_{\tau+1}=g(x_\tau)$ and $\Theta_{\tau+1}=f(\Theta_\tau)$, respectively. Since the invariant density is determined by the totality of generated sequences, h must also yield a relation between ϱ_f^* and ϱ_g^*. Using conservation of probability one may easily prove the relation

$$\varrho_g^*(x)=\varrho_f^*[h^{-1}(x)]\left|\frac{dh^{-1}(x)}{dx}\right|. \qquad (14)$$

Because of the one-to-one correspondence between sequences under g and f, ϱ_g^* is the asymptotic density for almost all $\{x_\tau\}$ if ϱ_f^* has this property with regard to $\{\Theta_\tau\}$. So the mere existence of a conjugating function h suffices to guarantee the existence and uniqueness of ϱ_g^* provided ϱ_f^* exists and is unique. h need not be a linear map. Thus g may be a function not satisfying the infimum condition (6) whereas f does so (Figure 2). Consequently the range of validity of T1 and T2 is extended to all dynamical laws which turn out to be conjugated to any f satisfying the propositions of T1 and T2 [17a].

Using the invariance of ϱ_g^* as well as the equivalence of ergodicity and the existence of a unique, invariant density which generates an absolutely continuous invariant measure, one may easily prove the following statement:

T3: Let f and g be dynamical laws conjugate by a one-to-one piecewise differentiable h. Then,

(a) if f is ergodic, g is so, too;

(b) if f is strongly mixing, so is g.

That means ergodicity and strong mixing are not characteristics of single dynamical laws, but properties of complete equivalence classes generated by conjugation.

Further we remark that Eq. (14) provides us with a possibility to construct dynamical laws g generating a required density ϱ_g^* by choosing an appropriate h to a given f with already known density ϱ_f^*. In practice one must, of course, restrict oneself to dynamical laws f whose densities have a sufficiently simple structure because of the implicit dependence of ϱ_f^* on x via h^{-1}. Figure 3 shows some examples of dynamical laws and their corresponding invariant densities found by conjugation.

Adler and Rivlin have proven the strongly mixing property of Tchebychev-polynomials [18]

$$T_p(x)=\cos(p \arccos x)\,, \quad p=2,3,4,\dots\,. \qquad (15)$$

These are conjugate to $N_p(\Theta)$ via $h(\Theta)=\cos(\pi\Theta)$. Therefore, according to T3, $N_p(\Theta)$, $p=2,3,4,\dots$ is strongly mixing, too.

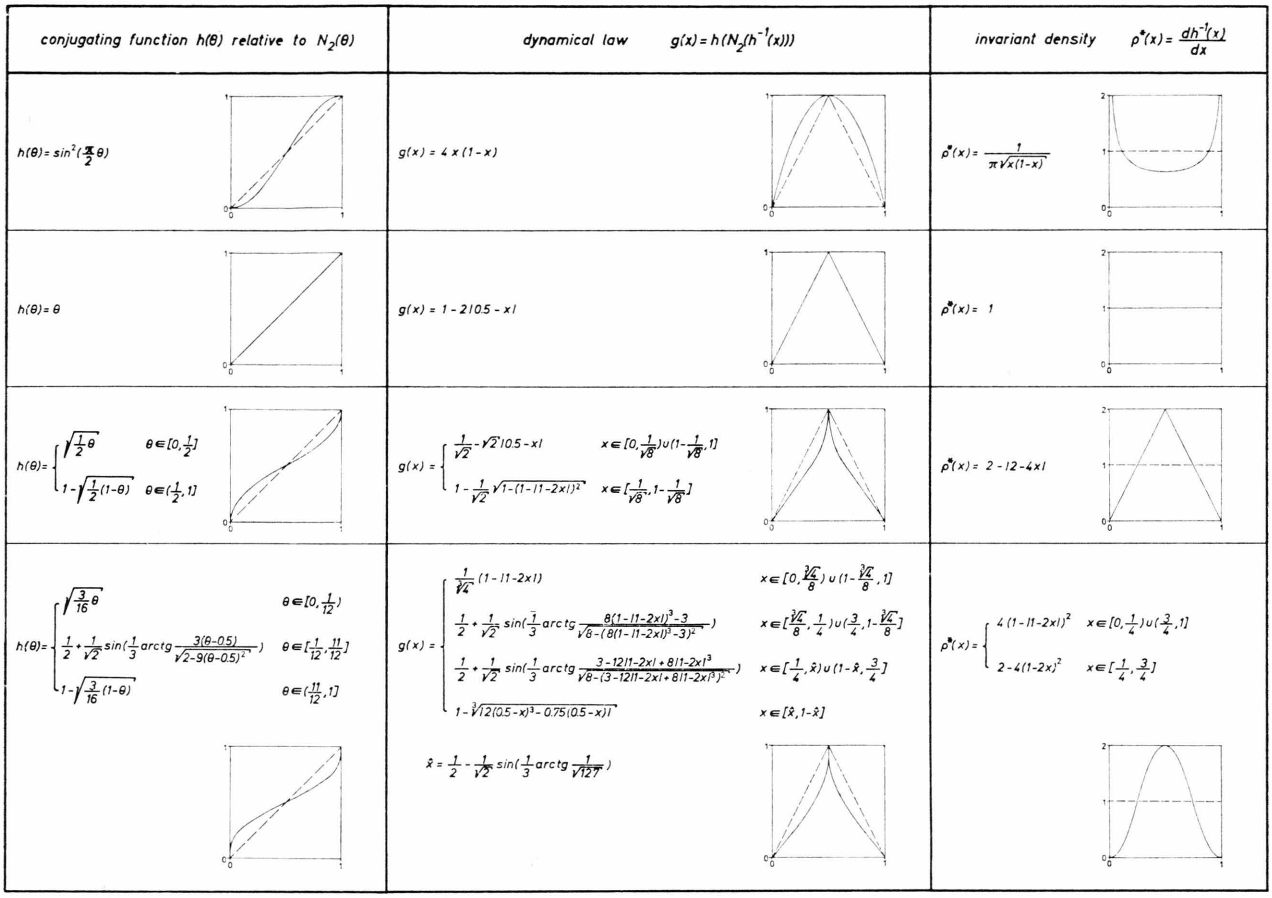

Fig. 3. Some dynamical laws, their invariant densities, and their conjugating functions relative to $N_2(\Theta)$. The second derivative of the dynamical law and of the respective invariant density have opposite signs. Slight changes in g may result in appreciable changes of ϱ_g^*.

V. Correlation Functions

We now study the time development of processes $\{x_\tau\}$ generated by a dynamical law f using the stationary correlation function (4). In this section we temporarily confine ourselves to ergodic laws f.

A process will be called δ-correlated if its correlation function has the structure

$$C_f(\tau) = C_f(0) \cdot \delta_{0\tau} \,. \tag{16}$$

All dynamical laws g: $[0,1] \to [0,1]$ which are conjugate to $N_{2n}(\Theta)$ with integer n and which have even symmetry in $[0,1]$ generate such processes. Proof: The conservation of symmetry requires the conjugating function $h(\Theta)$ to have odd symmetry relative to the point $[\frac{1}{2}, \frac{1}{2}]$; for examples see Figure 3. That means the derivative of its inverse is of even symmetry in $[0,1]$ and so is the transformed density $\varrho_g^*(x)$ according to (14). Using these properties of g and ϱ_g^* the correlation function $C_g(\tau)$ may be written in the form

$$C_g(\tau) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x \, g^{[\tau]}(x+\tfrac{1}{2}) \, \varrho_g^*(x+\tfrac{1}{2}) \, dx \,. \tag{17}$$

This integral vanishes for all $\tau = 1, 2, 3, \ldots$, and therefore the process generated by g must be δ-correlated. Another way to state the relation proven above is: The δ-correlation property of a process generated by a dynamical law with even symmetry is preserved under conjugation with odd symmetry. Notice: That does not mean δ-correlated processes can only be generated by symmetrical laws. The Tchebychev-polynomials $T_{2n+1}(x)$, $n = 1, 2, 3, \ldots$ are examples for antisymmetric laws generating δ-correlated processes. But this property in general is not preserved under conjugation with odd symmetry.

We now turn our attention to dynamical laws with less symmetry. To get a "feeling" for the changes occuring in the correlation function if there are deviations from even symmetry we consider a generalized version $\tilde{N}_p(x)$ of the broken linear transformation $N_p(x)$ of Section III. Let $\{y_\mu(\tau)\}$ with $y_{\mu-1}(\tau) < y_\mu(\tau)$ denote the set of points of discontinuity of $\tilde{N}_p{}^{[\tau]}$ and $[y_0(\tau), y_{p^\tau}(\tau)] = [0, 1]$. Then

$$\tilde{N}_p[x; \{y_\mu(1)\}] : \tag{18}$$
$$:= \tfrac{1}{2}\left(1 - (-1)^\mu \frac{2x - y_{\mu-1}(1) - y_\mu(1)}{y_\mu(1) - y_{\mu-1}(1)}\right)$$

if $x \in [y_{\mu-1}(1), y_\mu(1)]$; $\mu = 1, 2, \ldots, p$; $p \geqq 2$.

To calculate $C_{\tilde{N}_p}(\tau)$ we decompose the integration from 0 to 1 into subintegrals over $[y_{\mu-1}(\tau), y_\mu(\tau)]$, $\mu = 1, 2, \ldots, p$. Furthermore we use $\varrho^*_{\tilde{N}_p}(x) = 1$ which is easily concluded from the structure of the Frobenius-Perron operator.

$$\int_0^1 x \tilde{N}_p{}^{[\tau]}(x)\,dx$$
$$= \sum_{\mu=1}^{p^\tau} \int_{\frac{y_{\mu-1}(\tau)-y_\mu(\tau)}{2}}^{\frac{y_\mu(\tau)-y_{\mu-1}(\tau)}{2}} \left\{x + \frac{1}{2}[y_{\mu-1}(\tau) + y_\mu(\tau)]\right\}$$
$$\cdot \left\{\frac{1}{2} - (-1)^\mu \frac{x}{y_\mu(\tau) - y_{\mu-1}(\tau)}\right\} dx .$$

Thus

$$C_{\tilde{N}_p}(\tau) = \tfrac{1}{6}\sum_{\mu=1}^{p^\tau}(-1)^\mu y_\mu(\tau)\, y_{\mu-1}(\tau) - \tfrac{1}{12}(-1)^{(p^\tau)} . \tag{19}$$

An immediate consequence of (19) is

$$C_{\tilde{N}_p}(0) = 1/12 . \tag{20}$$

For $\tau = 1, 2, 3, \ldots$ a relation can be derived using the following equations which can be understood in terms of the geometric nature of the iteration process:

$$y_0(\tau) = 0, \quad y_{p^\tau}(\tau) = 1, \quad p = 2, 3, 4, \ldots, \quad \tau = 0, 1, 2, \ldots; \tag{21'}$$

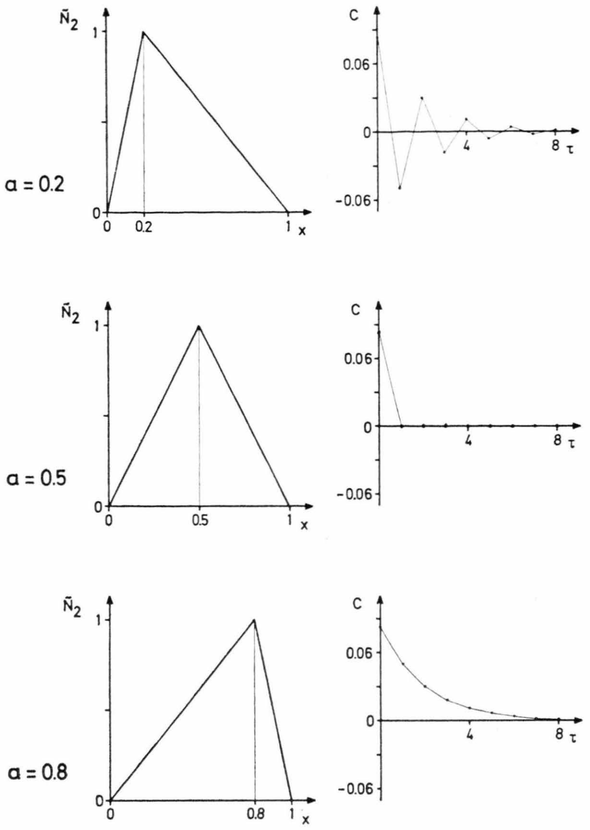

Fig. 4. $\tilde{N}_2(x; a)$ and the corresponding correlation functions for three different values of a. The skewness of $\tilde{N}_2$ destroys the δ-correlation and results in exponentially decaying monotone or oscillating correlations.

In general $\tilde{N}_p(x)$ yields an exponential decay of the time correlation with a characteristic time

$$\tau_c = -1/\ln\left|\sum_{\alpha=1}^{p}(-1)^{\alpha-1}\delta_\alpha{}^2\right| . \tag{24}$$

According to the sign of $\Sigma \ldots C_{\tilde{N}_p}(\tau)$ may be monotone or oscillating. If $\Sigma \ldots = 0$, $\tilde{N}_p$ generates a δ-correlated process. Figure 4 shows these three charac-

$$q := p^{\tau-1}, \quad \delta_\alpha := y_\alpha(1) - y_{\alpha-1}(1) ,$$

$$\left.\begin{aligned} y_{(\alpha-1)q+\mu}(\tau) &= \delta_\alpha y_\mu(\tau-1) + y_{\alpha-1}(1), & \alpha &= 1, 3, 5, \ldots \\ y_{(\alpha-1)q+\mu}(\tau) &= -\delta_\alpha y_{q-\mu}(\tau-1) + y_\alpha(1), & \alpha &= 2, 4, 6, \ldots \end{aligned}\right\} \quad \begin{aligned} &\tau = 1, 2, \ldots, \\ &\mu = 0, 1, 2, \ldots, q . \end{aligned} \tag{21''}$$

If (21″) is introduced in (19) a recursion formula for $C_{\tilde{N}_p}(\tau)$ results:

$$C_{\tilde{N}_p}(\tau) = \left(\sum_{\alpha=1}^{p}(-1)^{\alpha-1}\delta_\alpha{}^2\right) C_{\tilde{N}_p}(\tau-1) \begin{cases} p = 2, 3, 4, \ldots, \\ \tau = 1, 2, 3, \ldots, \end{cases} \tag{22}$$

$$C_{\tilde{N}_p}(\tau) = \frac{1}{12}\left(\sum_{\alpha=1}^{p}(-1)^{\alpha-1}\delta_\alpha{}^2\right)^\tau \qquad \begin{cases} p = 2, 3, 4, \ldots, \\ \tau = 0, 1, 2, \ldots . \end{cases} \tag{23}$$

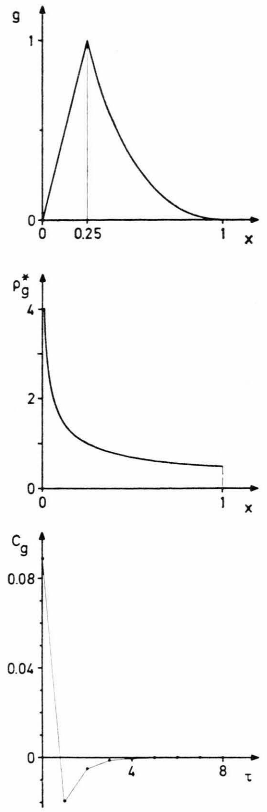

Fig. 5. The asymmetric dynamical law $g(x) = (1-2\,|\,0.5-\sqrt{x}\,|)^2$, its invariant density, and its correlation function.

teristic cases for $\tilde{N}_2(x;a)$. In particular we get:

$$\left.\begin{aligned} C_{N_{2p}}\ (\tau) &= \tfrac{1}{12}\,\delta_{0\tau} \\ C_{N_{2p+1}}(\tau) &= \tfrac{1}{12}(2\,p+1)^{-2\tau} \end{aligned}\right\} p = 1, 2, 3, \ldots \quad (25)$$

The generic feature revealed by this calculation is that any skewness in the dynamical law tends to destroy the δ-correlated character of the process. The characteristic time τ_c increases with the skewness. But, of course, there are exceptions, e.g. the already mentioned Tchebychev-polynomials whose special curvature just compensates the effect of skewness.

If we want to generate non-δ-correlated processes, we might do so by making use of our above observations and applying a non-antisymmetric conjugating function on a symmetric dynamical law:

$$f(\Theta) = N_2(\Theta)\,, \quad h(\Theta) = \Theta^2\,.$$

The conjugation yields (see also Fig. 5):

$$g(x) = (1-2\,|\,0.5-\sqrt{x}\,|)^2\,, \qquad (26)$$

$$\varrho_g^*(x) = 1/\sqrt{4\,x}\,, \qquad (27')$$

$$C_g(\tau) = \begin{cases} \frac{4}{45} & \tau = 0\,, \\ -\frac{7}{90}\left(\frac{1}{4}\right)^\tau & \tau = 1, 2, \ldots\,. \end{cases} \qquad (27'')$$

VI. A Misbehaved Conjugating Function

In the course of a synthesis of a dynamical law which generates processes with prescribed invariant density as well as prescribed correlation it might be valuable to have methods at hand that admit an independent adjustment of these two characteristics. We have already seen that conjugation with odd symmetry exclusively effects the density of symmetric laws but leaves the δ-correlation unchanged. Further we saw how the correlation functions of $\tilde{N}_p$ could be influenced by shifting their points of discontinuity whereas the density remained unchanged. Is this shifting procedure equivalent to a conjugation? We do not know the answer to this general problem. But the investigation of

$$\tilde{N}_2(x;a) = \begin{cases} x/a & x \in [0, a]\,, \\ (1-x)/(1-a) & x \in [a, 1]\,, \end{cases} \qquad (28)$$

reveals an obviously generic feature of the conjugating function

$$h_a(\Theta):\ \ \tilde{N}_2(x;a) = h_a(N_2[h_a^{-1}(x)])\,. \qquad (29)$$

There must be an h_a for all $a \in (0,1)$ since $\tilde{N}_2(x;a)$ can be generated by a series of infinitesimal shift operations which obviously cannot alter the totality of generated processes in its structure. So a one-to-one correspondence can be established.

We might start trying to find an h_a by mapping the left branch of $N_2(\Theta)$ on the left branch of $\tilde{N}_2(x;a)$. This is accomplished by

$$h_a(\Theta) = \Theta^{-\ln a/\ln 2}\,. \qquad (30)$$

But this cannot be the correct h_a because

(i) it does not take into account the right branches of both laws; and

(ii) the first derivative of h_a must, according to (14), be constant and equal to unity, since both laws generate a constant density.

So we have to try another method using intrinsic properties of N_2 as well as $\tilde{N}_2$. Both laws assign a unique predecessor to the point 1. Under $N_2(\Theta)$ the predecessor is $\frac{1}{2}$, under $\tilde{N}_2(x;a)$ it is a. We now generate sequences of predecessors for both laws and construct h_a by relating corresponding points to each other (see Figure 6). (It should be noticed that this method is of rather general character and in many cases yields a numerical access to the conjugating function.) If we write L for choosing the

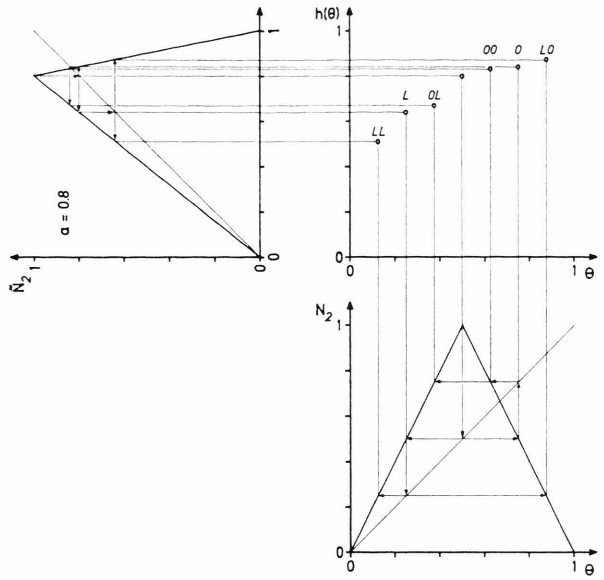

Fig. 6. Construction of a conjugating function $h(\Theta)$ by relating corresponding points of two dynamical laws to each other.

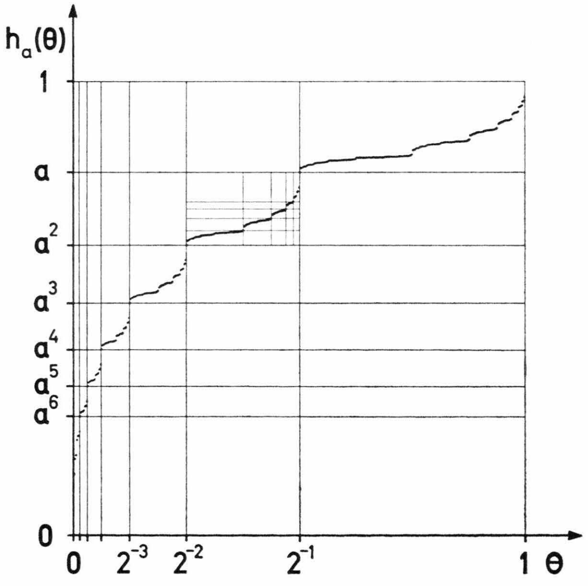

Fig. 7. The misbehaved conjugating function $h_a(\Theta)$ relating $N_2(\Theta)$ to $\tilde{N}_2(x; a)$ for $a=0.8$. The grid reveales the Cantor set structure of h_a.

predecessor on the left branch and 0 for choosing it on the right branch, each point is uniquely determined by its L0-pattern, which might be interpreted as a binary number. Figure 7 shows a result obtained this way by taking into account all points corresponding to binary numbers up to eight bits.

Is this solution compatible with the condition on the derivative of h_a? A closer inspection of this problem shows that all points constructed above are points of discontinuity where h_a is not differentiable. The set of these is not countable since the corresponding binary numbers take on all values in $(0, 1)$. h_a is an everywhere continuous and nowhere differentiable, strictly monotone function with Cantor set structure.

VII. Pure Chaos and Mixed States

Up to now we have considered ergodic dynamical laws. All iterates of these were ergodic, too. Physical systems described by this kind of laws are comparatively easy to deal with. Since ergodicity implies the instability of all periodic processes, which

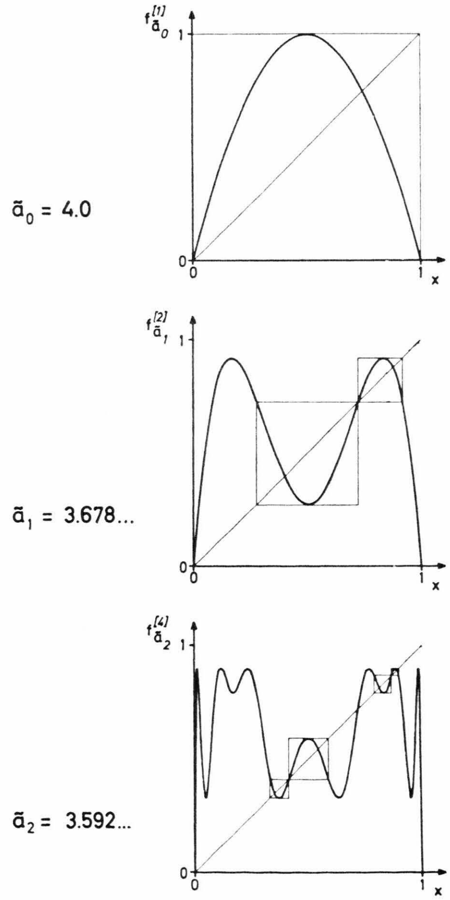

Fig. 8. Iterates of $f_a(x) = a\,x(1-x)$. If the parameter a takes on the critical value $\tilde{a}_\mu$, the iterate $f_a{}^{[2^\mu]}$ maps certain subintervals of $[0, 1]$ (indicated by squares) onto themselves.

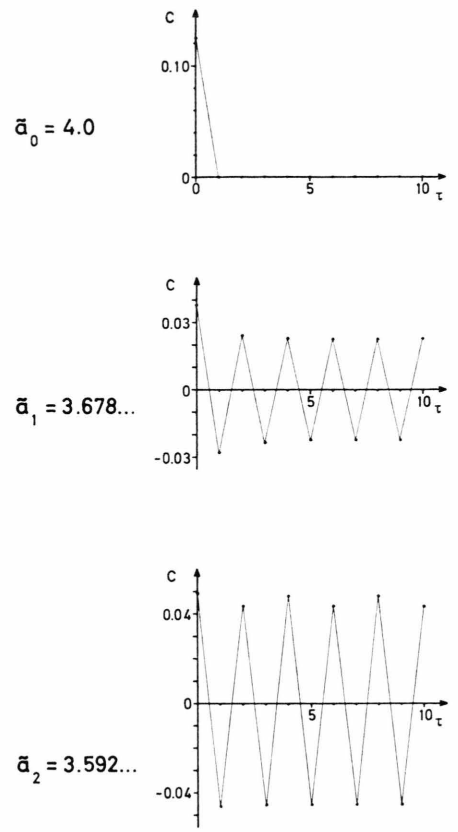

Fig. 9. Correlation functions of $f_a(x) = a\,x(1-x)$ for the critical values $\tilde{a}_\mu$, $\mu = 0, 1, 2$.

therefore become completely unimportant from the physical point of view, aperiodic behaviour prevails. If not only the dynamical law itself but also all of its iterates are ergodic, we call the system purely chaotic.

But there are also mixed states where periodic and chaotic time development are mingled with each other. In what follows we discuss these generic mixed states in the logistic equation [19]

$$f_a(x) = a\,x(1-x)\,, \quad a \in [1,4]\,. \tag{31}$$

If a is sufficiently small, f_a generates periodic processes which, as a is increased, change their period by repeated bifurcation from 1 to 2, 4, 8, ..., 2^n, ... There is a threshold $a_\infty \cong 3.5699$ of the parameter a where the limit 2^n, $n \to \infty$ for the periodicity is reached. F or $a = 4$ f_a and all its iterates are ergodic, i. e. f_4 generates pure chaos. Within $(a_\infty, 4)$ there are values of a where some iterates $f_a^{[\tau]}$ are decomposable despite the ergodicity of f_a itself. Figure 8 shows some examples of iterates at such values. Under these iterates the complete interval $[0, 1]$ obviously breaks up into several independent subintervals on which the respective iterate is ergodic. The dynamical law f_a periodically maps these subintervals onto each other and thus establishes a connection between them. Therefore the complete process can be understood as a superposition of a periodic motion and a chaotic motion in state space, which becomes particularly obvious by means of correlation functions. Figure 9 shows some of these corresponding to the cases depicted in Figure 8. The periodic part of the motion yields the oscillations whereas the chaotic part produces a decrease of the amplitudes to constant values. Since the subintervals mentioned above are comparatively small with respect to their relative distances this decrease is small in comparison to the oscillations.

To survey the solution set of the logistic equation we indicate these states of mixed and pure chaos by their invariant densities in a generalized bifurcation diagram (Figure 10). The gaps between the critical values $\tilde{a}_\mu$ might be filled by considering other periodic processes, e. g. periods three or five, and the corresponding states of mixed chaos.

A numerical analysis shows that the sequences $\{a_\mu\}$ and $\{\tilde{a}_\mu\}$ of bifurcation points approximately satisfy exponential laws:

$$\begin{aligned} a_\mu &\cong c_0 - c_1 \cdot e^{-c_2 \mu}\,; \\ c_0 &= 3.56994567 \pm 1.3 \cdot 10^{-7}\,, \\ c_1 &= 2.628 \pm 0.13\,, \\ c_2 &= 1.543 \pm 0.02\,, \end{aligned} \tag{32}$$

$$\begin{aligned} \tilde{a}_\mu &\cong \tilde{c}_0 + \tilde{c}_1 \cdot e^{-\tilde{c}_2 \mu}\,; \\ \tilde{c}_0 &= 3.56994565 \pm 1.3 \cdot 10^{-7}\,, \\ \tilde{c}_1 &= 2.152 \pm 0.06\,, \\ \tilde{c}_2 &= 1.530 \pm 0.03\,. \end{aligned} \tag{33}$$

Each sequence numerically converges to

$$a_\infty = 3.5699456\ldots \tag{34}$$

which therefore is the threshold of chaos.

VIII. Summarizing Discussion

We have studied the statistics arising from discrete, one-dimensional processes in terms of the invariant asymptotic distributions and of the stationary correlations between successive events during

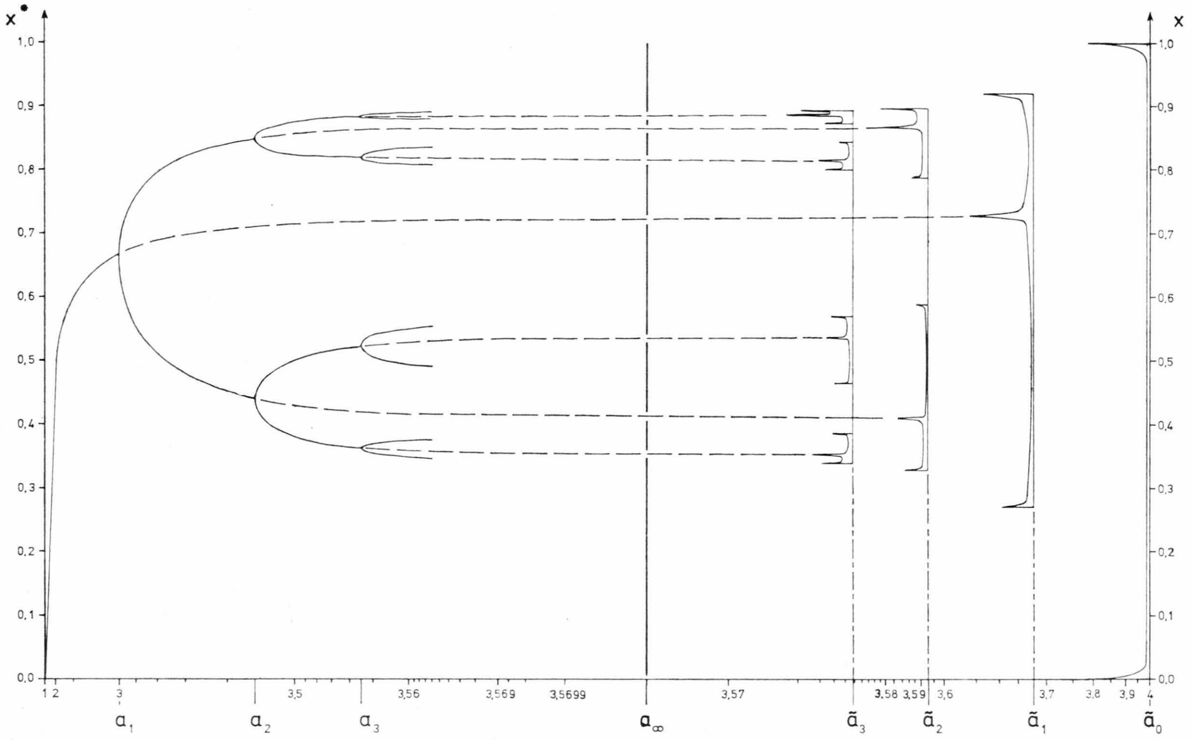

Fig. 10. Generalized bifurcation diagram for $f_a(x) = a\,x(1-x)$. The chaos states at $\tilde{a}_\mu$ are indicated by their invariant densities. [To achieve a better graphical representation a nonlinear scale was used on the abscissa: $a' = \sqrt[5]{\tanh(a-a_\infty)}$.]

the process. Following Rössler's [20] considerations of chaotic degrees of freedom one might relate this kind of process to multi-dimensional continuous systems. The results reported may shed some light on the remarkable fact that simple deterministic processes may physically appear as being chaotic. Chaos needs a probabilistic description. It then is necessary to derive the connection between the dynamical process and its probability distribution or correlations. In a conservative $N \sim 10^{23}$ particle system this connection is: Hamilton's equations of motion $\rightarrow$ canonical distribution. In a $N = 1$ degree of freedom system, being discrete in time, the form of the dynamical law determines the probabilistic properties much more detailed, although general concepts like ergodicity apply as well. It is this implication of a given nonlinear deterministic process on its corresponding probability distribution function which lies at the basis of many classical problems as for example the statistical theory of turbulence. The occurence of mixed states reveals a kind of steady transition from purely periodic to purely chaotic behaviour. It will probably be very difficult to get experimentally hold of these mixed states since they require a very precise setting of test parameters.

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